WELFEP: a Round Robin for Weakest-Link Finite Element Postprocessors

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(Received 16 January 1992; revised version received 24 February 1992; accepted 22 April 1992)

Abstract

A numerical round robin for finite element postprocessors for weakest-link failure probability predictions has been analysed. Three selected problems were analysed by the participants using a number of different postprocessors. In general, good agreement was obtained for the predicted mean nominal failure stress. The predicted values for the failure probability show much more scatter due to the sensitivity of this quantity to integration errors.

Ein numerischer Sammelbrief für Finite-Elemente Postprozessoren zur Vorhersage der Versagenswahrscheinlichkeit schwächster Bindungen wurde analysiert. Drei ausgewählte Probleme wurden von den Teilnehmern mit Hilfe verschiedener Postprozessoren analysiert. Im Allgemeinen wurde eine gute Übereinstimmung der vorhergesagten mittleren nominalen Bruchspannung erreicht. Die vorhergesagten Werte der Bruchwahrscheinlichkeit dagegen zeigen eine weitaus stärkere Streuung aufgrund der Empfindlichkeit dieser Größe in Bezug auf Integrationsfehler.

Cet article relate les résultats d'un test interlaboratoire des postprocesseurs par éléments finis, relatif à la prédiction de la probabilité de rupture au chaînon le plus faible de composants. Trois problèmes ont été sélectionnés puis analysés par les participants en utilisant un certain nombre de postprocesseurs. En général, on a obtenu un bon accord quant à la prédiction de la valeur nominale moyenne de rupture. Par contre, la probabilité de rupture est beaucoup plus dispersée, en raison de la sensibilité de cette grandeur aux erreurs d'intégration.

1 Introduction

In recent years the description and prediction of the mechanical behaviour of modern ceramic materials has been the subject of a number of analyses. Brittle fracture and slow crack growth have been analysed both experimentally and theoretically with the Weibull statistic distribution used to model the statistical distribution of strength of these materials. Experimental data are gathered from a variety of test methods, not only for material characterization but also for design purposes. Various methods have been proposed to transfer data from one test to another or to use these test data to make strength predictions for more complicated situations which give rise to multiaxial stress systems. To use these mathematical models in a complex body it is usually necessary to use a finite element program. The mathematical models for the failure probability require an integral of stress over a volume or surface and for this purpose a number of postprocessors have been developed to allow data from the finite element program to be used to evaluate the failure probability.[1–5]

In 1990 an informal working group was formed to

Journal of the European Ceramic Society 0955-2219/93/$6.00

discuss these postprocessors. The working group was named WELFEP, which stands for Weakest-Link Failure Probability Prediction by Finite Element Postprocessors. After a first meeting in September 1990 it was decided to start a common activity, initially focusing on two topics:

—Gathering information on the design and structure of the various postprocessors by means of a questionnaire.
—Comparison of predictions of the postprocessors for a number of selected problems (numerical round robin).

The results of these activities were discussed in April 1991 and have been presented in an internal report. Also in 1991 the working group became part of Technical Committee 6 'Technical Ceramics' of the European Structural Integrity Society (ESIS). Within the working group it was felt that the results of these first common activities may provide valuable information for a broader audience. Therefore this paper is an attempt to give an overview of the findings of the questionnaire and of the round robin.

2 Theoretical Background

In all the postprocessors employed in the analyses the key quantity to be evaluated is the probability of failure P_f according to weakest-link statistics of a component containing volume and/or surface flaws. A number of models have been proposed in literature on how P_f can be calculated for a given geometry, stress state and fracture criterion. Among many others, the models of Weibull,[6] Freudenthal[7] and Batdorf & Heinisch[8] are frequently mentioned. Apart from whether or not a particular model is valid for a certain material in a certain application, there is generally little consistency in the way the various models are formulated. The formulation adopted by the working group is given by[6–10]

$$P_f^V = 1 - \exp\left[-\left(\frac{1}{m}!\right)^m \left(\frac{S_{nom}}{S_u}\right)^m \frac{V}{V_u}\Sigma(V)\right] \quad (1)$$

for volume flaws and

$$P_f^A = 1 - \exp\left[-\left(\frac{1}{m}!\right)^m \left(\frac{S_{nom}}{S_u}\right)^m \frac{A}{A_u}\Sigma(A)\right] \quad (2)$$

for surface flaws, where

$$\Sigma(V) = \frac{1}{V}\int_V \frac{1}{4\pi}\int_{B_u}\left(\frac{\sigma_{eq}}{S_{nom}}\right)^m dB_u\, dV \quad (3)$$

$$\Sigma(A) = \frac{1}{A}\int_A \frac{1}{2\pi}\int_{C_u}\left(\frac{\sigma_{eq}}{S_{nom}}\right)^m dC_u\, dA \quad (4)$$

with

V = volume of component
A = surface of component
m = Weibull modulus
S_{nom} = a given nominal or reference stress
S_u = strength per unit volume V_u or unit surface A_u
V_u = unit volume
A_u = unit surface area
$\Sigma(V)$ = stress volume integral
$\Sigma(A)$ = stress surface integral
σ_{eq} = equivalent fracture stress
B_u = surface of a sphere with unit radius
C_u = circle with unit radius
$(1/m)! = \Gamma[1+(1/m)]$ with Γ as the gamma function

If the normal stress σ_n and the shear stress τ acting on a single crack are defined by

$$\sigma_n = \mathbf{n}\cdot\sigma\cdot\mathbf{n} \quad (5)$$

$$\tau^2 = \mathbf{n}\cdot\sigma^2\cdot\mathbf{n} - \sigma_n^2 \quad (6)$$

with σ as the stress tensor and $\mathbf{n}$ as the normal vector to C_u or B_u, some specific models frequently mentioned in literature can be given as:

—Mode I failure or normal stress averaging (NSA):[6]

$$\sigma_{eq} = \begin{cases}\sigma_n & \text{if } \sigma_n > 0\\ 0 & \text{if } \sigma_n \le 0\end{cases} \quad (7)$$

—Maximum non-coplanar strain energy release rate (GMAX):[11]

$$\sigma_{eq}^4 = \begin{cases}\sigma_n^4 + 6\sigma_n^2\tau^2 + \tau^4 & \text{if } \sigma_n > 0\\ 0 & \text{if } \sigma_n \le 0\end{cases} \quad (8)$$

Implementation of eqn (8) in eqns (1) to (4) leads to the model proposed by Lamon & Evans[12] (multiaxial elemental strength model—MuESt).

—Principle of independent action (PIA):[7]

$$\sigma_{eq}^m = \langle S_1\rangle^m + \langle S_2\rangle^m + \langle S_3\rangle^m \quad (9)$$

Here $S_1 \ge S_2 \ge S_3$ are the principal stresses derived from the stress tensor σ and

$$\langle x\rangle = \begin{cases}x & \text{if } x > 0\\ 0 & \text{if } x \le 0\end{cases}$$

In the case where $\sigma_n \le 0$, alternative formulations have been proposed in which σ_{eq} is obtained from a weighted combination of the normal and shear stress acting on the crack.[13]

3 Numerical Aspects

In calculating P_f some essential input like the appropriate fracture criterion and parameters such as m, S_u, V_u and S_{nom} are needed. It must be mentioned that S_u and V_u (A_u) are not independent variables, as S_u gives the strength per unit volume V_u (A_u) in the case of uniform triaxial (biaxial) tension. Then, in most postprocessors, P_f is calculated using the intermediate results for $\Sigma(V)$ or $\Sigma(A)$ (or a related quantity). The integration with respect to V or A is done by summing over all appropriate elements in the FE mesh. For sake of briefness and confining the discussion to volume flaws, $\Sigma(V)$ can then be obtained from

$$\Sigma(V)=\sum_{i=1}^{N_v}\frac{V_i}{V}\Sigma(V_i) \quad (10)$$

where N_v is the number of relevant volume elements and $\Sigma(V_i)$ is the stress volume integral for element i which is obtained from relation (3):

$$\Sigma(V_i)=\frac{1}{V_i}\int_{V_i}F_i\,\mathrm{d}V_i \quad (11)$$

with F_i given by

$$F_i=\frac{1}{4\pi}\int_{B_u}\left(\frac{\sigma_{eq}}{S_{nom}}\right)^m \mathrm{d}B_u \quad (12)$$

The integration over B_u (or C_u in case of surface flaws) is also referred to as orientation integration.

Now, in the postprocessors considered, eqn (11) is evaluated either by subelement integration or by Gauss–Legendre integration:

—Subelement integration:

With subelement integration the elements are divided into N_s subelements in which the stresses are assumed constant. Then, for each subelement k, eqn (12) can be evaluated to yield F_i^k. If the volume of the subelement equals V_i^k then eqn (11) yields

$$\Sigma(V_i)=\frac{1}{V_i}\sum_{k=1}^{N_s}F_i^k V_i^k \quad (13)$$

—Gauss–Legendre integration:

For Gauss–Legendre integration the stresses are sampled at N_g Gauss points. Then, for each Gauss point k, F_i^k can be determined and combined to yield

$$\Sigma(V_i)=\frac{1}{V_i}\sum_{k=1}^{N_g}F_i^k w_i^k \quad (14)$$

with w_i^k as the appropriate weight factor for integration point k.

In the postprocessors considered F_i^k is evaluated by two-dimensional Gauss–Legendre integration or by two-dimensional midpoint-rule integration.

The procedure outlined has been implemented in the various postprocessors in a more or less similar manner. One of the goals of the round robin was to see whether differences in the implementation yield comparable results. It was known beforehand that integration errors can lead to large errors in P_f because, for larger values of m, σ_{eq}^m can vary strongly. A measure which is not so sensitive to integration errors is given by the predicted mean value for S_{nom}:

$$\bar{S}_{nom}=S_u\left[\frac{V_u}{V\Sigma(V)}\right]^{1/m} \quad \text{for volume flaws}$$
$$\bar{S}_{nom}=S_u\left[\frac{A_u}{A\Sigma(A)}\right]^{1/m} \quad \text{for surface flaws} \quad (15)$$

Then eqns (1) and (2) yield

$$P_f=1-\exp\left[-\left(\frac{1}{m}!\right)^m\left(\frac{S_{nom}}{\bar{S}_{nom}}\right)^m\right] \quad (16)$$

If the error ε_P in the predicted failure probability is defined by

$$\varepsilon_P=\frac{P_{f,calc}-P_{f,exact}}{P_{f,exact}} \quad (17)$$

and the error ε_s in the predicted value for $\bar{S}_{nom}$ by

$$\varepsilon_s=\frac{\bar{S}_{nom,calc}-\bar{S}_{nom,exact}}{\bar{S}_{nom,exact}} \quad (18)$$

eqn (16) can be used to relate ε_P and ε_s in case of small ε_s. Using a Taylor expansion one obtains

$$\varepsilon_P=\frac{1-P_{f,calc}}{P_{f,calc}}\ln(1-P_{f,calc})m\varepsilon_s \quad (19)$$

Now, for small failure probabilities, eqn (19) can be written as

$$\varepsilon_P=-m\varepsilon_s \quad \text{for small } \varepsilon_s \text{ and } P_f \quad (20)$$

Equation (20) shows that ε_s has to be small for larger values of m to give a reliable prediction for P_f. Therefore an accurate integration scheme is required to deal with higher values of m.

4 Results of the Questionnaire

The questionnaire circulated contained questions in four categories:

- —General information with respect to the development of the postprocessor.
- —Characteristics of the postprocessor.
- —Special information about calculation methods.
- —Experiences/problems.

The questionnaire was completed by ten participants representing eight different postprocessors.

4.1 General information

The results show that all postprocessors are written in Fortran. The postprocessors used are to be interfaced with an FE program in order to obtain the necessary element data and stresses. Some postprocessors have an interface to one FE program, others to two or more, although no universal interface seems to be present. A variety of FE packages have been mentioned: ABAQUS, ADINA, ANSYS, MFIELD, NASTRAN, PAFEC and SYSTUS. The programs run on a variety of micro- and supercomputers.

4.2 Characteristics of the postprocessor

Most postprocessors support fracture criteria such as normal stress averaging (eqn (7)) and principle of independent action (eqn (9)), although some provide several other fracture criteria as well. In most cases 2D and 3D structures can be analysed with both volume and surface flaws. The type of elements supported are generally 2D isoparametric plane stress elements with 4 or 8 nodes and 3D isoparametric brick elements with 8 or 20 nodes. Some programs support much more types of 2D and 3D elements.

4.3 Calculation methods

As mentioned in Section 3, either subelement or Gauss–Legendre integration is used for surface/volume integration. In the case of subelement integration for 2D elements $3*3$ or $4*4$ and for 3D elements $3*3*3$ or $4*4*4$ subelements are used. For Gauss–Legendre integration a four-noded 2D and an eight-noded 3D element with 1 Gauss point (centroidal value), an eight-noded 2D element with 4 Gauss points and a 20-noded 3D element with 8 Gauss points are used. Some programs sample at more Gauss points to increase the accuracy of the integration process.

To carry out the orientation integration to determine the factors F_i^k mentioned in Section 3 (eqn (12)) generally Gauss–Legendre integration is applied with 10, 15 or 20 Gauss points for integration over C_u and $10*10$, $15*15$ or $20*20$ Gauss points for integration over B_u.

To be able to identify the surface of a component, which is then used in a surface flaw analysis, several techniques are employed. One is to add a special surface mesh (shell or beam elements with zero stiffness) in the FE calculations which does not contribute to the stiffness of the component. Then the FE package will be able to calculate the stresses in these surface elements which subsequently can be used in the postprocessor. This procedure implies that the FE calculation will be more time consuming. An alternative is that the postprocessor contains an algorithm to identify the surface of a volume mesh and to extrapolate the stresses from the interior to the surface.

Compressive principal stresses are treated in some different manners depending on the fracture criterion. In the case of PIA (eqn (9)) they are either neglected or weighted with a negative weight factor. In the case of NSA (eqn (7)) or GMAX (eqn (8)) negative principal stresses may exist as long as the normal stress σ_n is positive. If σ_n is negative either σ_n or σ_{eq} is set to zero.

4.4 Experiences/problems

Most programs can be run interactively or use a special input file. Some programs are highly modular and can be modified relatively easily to incorporate other fracture criteria or element types, while others are more dedicated and cannot be modified without expert knowledge.

Most participants did some tests with the program to assure correctness of the code, but also stated that a finer mesh than normally would be used for a conventional FE analysis is needed in the case of steep stress gradients and/or a high value of the Weibull modulus m. Ideally, mesh adaptation for use in the postprocessor is not necessary if the mesh is sufficiently refined to yield sufficiently accurate stress predictions. In practice, however, a sensitivity analysis based on mesh refinement is almost unavoidable to assure correctness of the results. In general, it can be stated that the larger the number of subelements or Gauss points for the integration with respect to V or A, the less mesh refinement is needed.

5 Results of the Numerical Round Robin

In the numerical round robin three test cases were analysed by the participants. In developing these test cases it was recognized that any influence of the FE package should be ruled out. Therefore problems were chosen such that it could be expected that any FE package used would yield the same stresses throughout the component. Isotropic linear elastic material behaviour was chosen. Test case 1 deals with pure bending of a beam in which the displacements at the ends of the beam are prescribed as shown in Fig. 1. In this case analytical solutions can easily be obtained. Test case 2 (Fig. 2) deals with four-point bending of a notched beam for which no

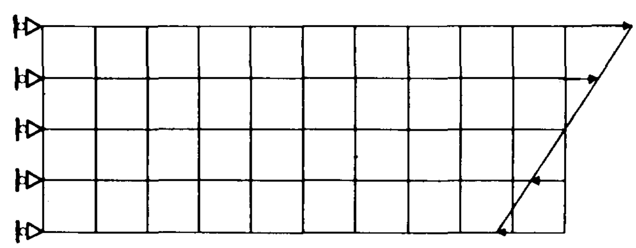

Fig. 1. Finite element mesh and boundary conditions for WELFEP test case 1: pure bending of a beam by prescribed displacements (plane stress analysis).

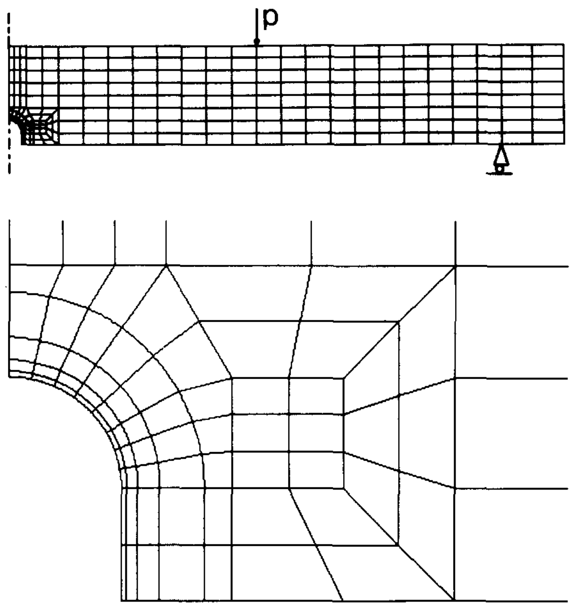

Fig. 2. Finite element mesh for WELFEP test case 2: four-point bending of a notched beam (plane stress analysis). The lower picture zooms in on the finite element mesh near the notch.

analytical solution is available. Test case 3 (Fig. 3) deals with pure torsion of a tube in which one end is clamped and on which the displacements at the other end are prescribed. In this case an analytical solution can also be given.

In all test cases a standard mesh was given, together with values for parameters, as E, ν, S_u, V_u, S_{nom} and m. Optionally other meshes or element types could be analysed. Of particular interest was the influence of m as it was expected that for larger values of m the results are more sensitive to integration errors (see also Section 3). The results can now be analysed in two ways: either by looking at differences/errors in the predicted value for the mean nominal stress $\bar{S}_{nom}$ or by looking at differences/errors in P_f. From the analysis given in Section 3 it is clear that these are related and that small errors in $\bar{S}_{nom}$ are propagated by about a factor $-m$ in errors in P_f. Therefore here only attention will be paid to errors in $\bar{S}_{nom}$.

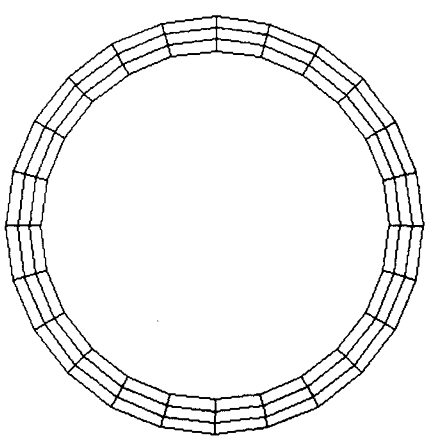

Fig. 3. Finite element mesh for WELFEP test case 3: pure torsion of a tube. The mesh contains one element along the length of the tube (3D stress analysis).

In elaborating the results obtained by the various participants the need for a standardized notation and parameter interpretation became clear. Strongly different results were reported, mainly due to translation of the unit strength S_u into an appropriate scale factor for the specific postprocessor. When straightforward corrections were applied the results became well comparable. Analysis of the results revealed that those postprocessors that use $4*4$ or $4*4*4$ subelement or Gauss integration tend to yield more accurate results than those which apply $3*3/3*3*3$ subelement or Gauss–Legendre integration (the latter yield comparable results). This can be illustrated by considering Figs 4 and 5, which show the errors in the predicted values for $\bar{S}_{nom}$ in the case of volume flaws for cases 1 and 3 for $m=25$. Postprocessors p1, p2, p3 and p4 all apply $3*3$ subelement integration or Gauss–Legendre integration, whereas p5, p6, p7 and p8 use $4*4$ subelement or Gauss integration. In general, however, the differences/errors are limited. The errors shown in Fig. 4 were not quite representative, as for test cases 2 and 3 the differences/errors are only about 1%. This also leads to the conclusion that the mesh used in test case 1 (Fig. 1) for a number of postprocessors is not fine enough. An important observation is that in almost every case $\bar{S}_{nom}$ is overestimated, which also means that P_f is systematically underestimated. However, this is probably due to the fact that the

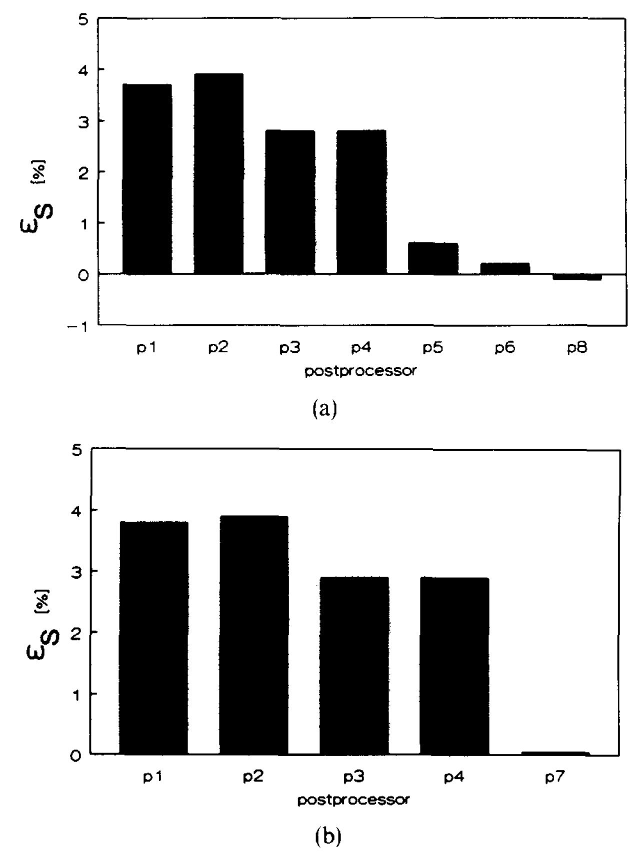

Fig. 4. Errors ε_s for test case 1 and $m=25$. p1, p2, etc., are different postprocessors. (a) PIA = principle of independent action (eqn (9)); (b) NSA = normal stress averaging (eqn (7)).

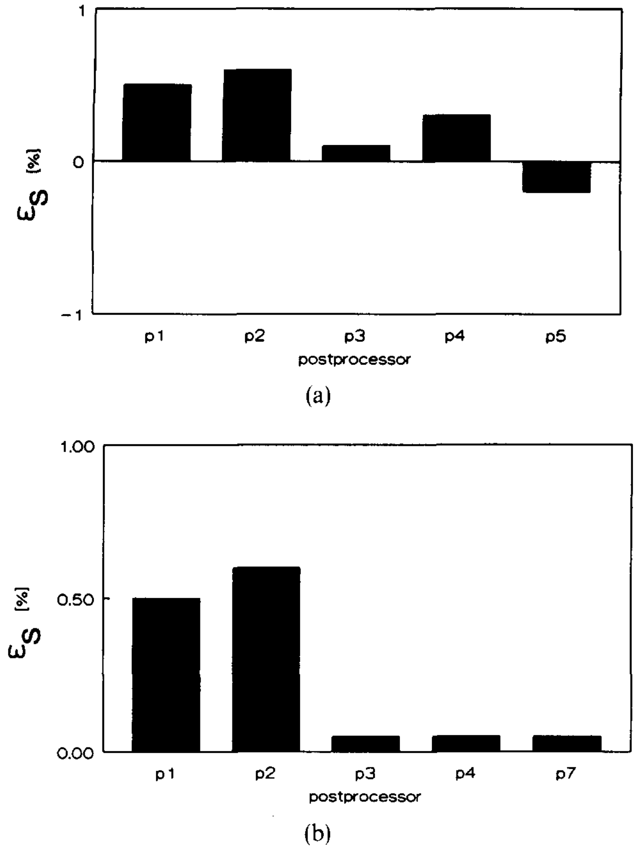

Fig. 5. Errors ε_s for test case 3 and $m = 25$. p1, p2, etc., are different postprocessors. (a) PIA = principle of independent action (eqn (9)); (b) NSA = normal stress averaging (eqn (7)).

stresses are monotonically decreasing in all test cases.

6 Concluding Remarks

An evaluation has been made of features and results for selected problems of a number of postprocessors for weakest-link failure probability calculations. From the results of the round robin calculations an indication about the differences between various postprocessors can be obtained. Caution is necessary in the case of steep stress gradients and/or high Weibull moduli, as in these cases a finer mesh may be required. The need for a standardized notation of the failure probability relation has become clear as the different programs contain a different interpretation of the scale factor.

Within the working group it has been decided to continue the common activity by analysis of some additional problems in which stress multiaxiality plays an important role. Then the influence of the choice of the fracture criterion becomes much more pronounced as shear stresses become important. Apart from this the postprocessors can also be used for lifetime prediction under slow crack growth and for analysis of proof-testing. These topics are likely to become part of continued collaboration within the working group.

Acknowledgement

The authors wish to thank all participants in the WELFEP working group for fruitful discussions and for providing information on a non-sponsored basis.

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